Infinitely many maximal primitive positive clones in a diagonalizable algebra

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Abstract. We present a rather simple example of infinitely many maximal primitive positive clones in a diagonalizable algebra, which serve as an algebraic model for the provability propositional logic GL.

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1 Introduction

The present paper deals with clones of operations of a diagonalizable algebra which are closed under definitions by existentially quantified systems of equations. Such clones are called primitive positive clones [1] (in [2] they are referred to as clones acting bicentrally, and are also called parametrically closed classes in [3, 4]). Diagonalizable algebras [5] are known to be algebraic models for the propositional provability logic GL [6].

The proof that there are finitely many primitive positive clones in any $k$-valued logic was given in [1]. In the case of 2-valued boolean functions, i.e. $\text{card}(A) = 2$, A.V. Kuznetsov stated there are 25 primitive positive clones [3], and A.F. Danil’čenko proved there are 2986 primitive positive clones among 3-valued functions [4]. In the present paper we construct a diagonalizable algebra, generated by its least element, which has infinitely many primitive positive clones, moreover, these primitive positive clones are maximal.

2 Definitions and notations

Diagonalizable algebras. A diagonalizable algebra [5] $\mathfrak{A}$ is a boolean algebra $\mathfrak{A} = (A; \&, V, \supset, \neg, 0, 1)$ with an additional operator $\Delta$ satisfying the following relations:

\[
\Delta(x \supset y) \leq \Delta x \supset \Delta y,
\]

\[
\Delta x \leq \Delta \Delta x,
\]

\[
\Delta(\Delta x \supset x) = \Delta x,
\]

\[
\Delta 1 = 1,
\]

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where 1 is the unit of \( \mathcal{M} \).

We consider the diagonalizable algebra \( \mathcal{M} = (M; \&., \lor, \circ, \neg, \Delta) \) of all infinite binary sequences of the form \( \alpha = (\mu_1, \mu_2, \ldots) \), \( \mu_i \in \{0, 1\} \), \( i = 1, 2, \ldots \). The boolean operations \( \&., \lor, \circ, \neg, \Delta \) over elements of \( M \) are defined component-wise, and the operation \( \Delta \) over element \( \alpha \) is defined by the equality \( \Delta \alpha = (1, \nu_1, \nu_2, \ldots) \), where \( \nu_i = \mu_1 \& \cdots \& \mu_i \). Let \( \mathcal{M}^\ast \) be the subalgebra of \( \mathcal{M} \) generated by its zero \( 0 \) element \( (0, 0, \ldots) \). Remark the unit of \( \mathcal{M}^\ast \) is the element \( (1, 1, \ldots) \).

As usual, we denote by \( x \sim y \) and \( \Delta^2 x, \ldots, \Delta^{n+1} x, \ldots \) the corresponding functions \( (\neg x \lor y) \& (\neg y \lor x) \) and \( \Delta \Delta x, \ldots, \Delta \Delta^a x, \ldots \). Denote by \( \square x \) the function \( x \& \Delta x \) and denote by \( \nabla x \) the function \( \neg \square \neg \square x \).

**Primitive positive clones.** The term algebra \( T(\mathfrak{D}) \) of \( \mathfrak{D} \) is defined as usual, stating from constants \( 0, 1 \) and variables and using operations \( \&., \lor, \circ, \neg, \Delta \). We consider the set \( T \) of all term operations of \( \mathcal{M}^\ast \), which obviously forms a clone \([7]\).

Let us recall that a **primitive positive formula** \( \Phi \) over a set of operations \( \Sigma \) of \( \mathfrak{D} \) is of the form

\[
\Phi(x_1, \ldots, x_m) = (\exists x_{m+1}) \cdots (\exists x_n)((f_1 = g_1) \& \cdots \& (f_s = g_s)),
\]

where \( f_1, g_1, \ldots, f_s, g_s \in T(\mathfrak{D}) \cup Id_A \) and the formula \((f_1 = g_1) \& \cdots \& (f_s = g_s)\) contains variables only from \( x_1, \ldots, x_n \). An \( n \)-ary term operation \( f \) of \( T(\mathfrak{D}) \) is **(primitive positive) definable over** \( \Sigma \) if there is a primitive positive formula \( \Phi(x_1, \ldots, x_n, y) \) over \( \Sigma \) of \( T(\mathfrak{D}) \) such that for any \( a_1, \ldots, a_n, b \in \mathfrak{D} \) we have \( f(a_1, \ldots, a_n) = b \) if and only if \( \Phi(a_1, \ldots, a_n, b) \) on \( \mathfrak{D} \) \([8]\). Denote by \( [\Sigma] \) all term operations of \( \mathfrak{D} \) which are primitive positive definable over \( \Sigma \) of \( \mathfrak{D} \). They say also \( [\Sigma] \) is a primitive positive clone on \( \mathfrak{D} \) generated by \( \Sigma \). If \( [\Sigma] \) contains \( T(\mathfrak{D}) \) then it is referred to as a **complete primitive positive clone on** \( \mathfrak{D} \). A primitive positive clone \( C \) of \( \mathfrak{D} \) is **maximal in** \( \mathfrak{D} \) if \( T(\mathfrak{D}) \not\subseteq C \) and for any \( f \in T(\mathfrak{D}) \setminus C \) we have \( T(\mathfrak{D}) \subseteq [C \cup \{f\}] \).

Let \( \alpha \in \mathfrak{D} \). They say \( f(x_1, \ldots, x_n) \in T(\mathfrak{D}) \) conserves the relation \( x = \alpha \) on \( \mathfrak{D} \) if \( f(\alpha, \ldots, \alpha) = \alpha \). According to \([9]\) the set of all functions that preserves the relation \( x = \alpha \) on an arbitrary \( k \)-element set is a primitive positive clone.

### 3 Preliminary results

We start by presenting some useful properties of the term operations \( \Delta, \square \) and \( \nabla \) of \( \mathcal{M}^\ast \).

**Proposition 1.** Let \( x, y \) be arbitrary elements of \( \mathcal{M}^\ast \). Then:

1. \( \square x \geq \Delta 0 \) if and only if \( \nabla x = 1 \)
2. \( \square x = 0 \) if and only if \( \nabla x = 0 \)
3. For any \( x, y \), either \( \square x \leq \square y \) or \( \square y \leq \square x \)
4. \( \Delta x = \Delta \square x \)
5. \( \nabla 0 = 0, \nabla 1 = 1 \)
6. \( \square x \geq \Delta 0 \) if and only if \( \square \neg x = 0 \)
\[ □x = 0 \text{ if and only if } □\neg x \geq \Delta 0 \]  
(7)

**Proof.** The proof is almost obvious by construction of the algebra \( \mathcal{M}^* \). □

Let us mention the following.

**Remark 1.** Any function \( f \) of \( T(\mathfrak{D}) \) is primitive positive definable on \( \mathfrak{D} \) via the system of functions \( x \& y, x \lor y, x \rightarrow y, \neg x, \nabla y \).

Let us consider on \( \mathfrak{D} \) the following functions (8) and (9) of \( T(\mathfrak{D}) \), denoted by \( f_-(x, y) \) and \( f_\Delta(x, y) \) correspondingly, where \( \alpha, \beta \in \mathfrak{D} \), \( \alpha_i = \neg \Delta \xi \), where \( \xi \neq \alpha_i \) and \( \eta \neq \alpha_i ;

\[
(\nabla \neg(x \sim y) \& ((\neg x \sim y) \sim \xi)) \lor (\nabla(x \sim y) \& \alpha_i),
\]
(8)
\[
(\nabla y \& ((\Delta x \sim y) \sim \eta)) \lor (\nabla y \& \alpha_i).
\]
(9)

**Proposition 2.** Let arbitrary \( \alpha, \beta \in \mathcal{M}^* \). If \( \neg \alpha = \beta \) on \( \mathcal{M}^* \), then

\[
f_-(\alpha, \beta) = \xi
\]
on \( \mathcal{M}^* \).

**Proof.** Since \( \neg \alpha = \beta \) we get \( \alpha \sim \beta = 0, \neg(\alpha \sim \beta) = 1 \) and by (5) we have

\[
\nabla(\alpha \sim \beta) = 0, \nabla \neg(\alpha \sim \beta) = 1,
\]
which implies

\[
f_-(\alpha, \beta) = (1 \& (1 \sim \xi)) \lor (0 \& \alpha_i) = \xi.
\]
□

**Proposition 3.** Let arbitrary \( \alpha, \beta \in \mathcal{M}^* \). If \( \neg \alpha \neq \beta \) on \( \mathcal{M}^* \), then

\[
f_-(\alpha, \beta) \neq \xi
\]
on \( \mathcal{M}^* \).

**Proof.** Since \( \neg \alpha \neq \beta \) we get \( \neg \alpha \sim \beta \neq 1, \alpha \sim \beta \neq 0 \). We distinguish two cases:

1) \( □(\alpha \sim \beta) = 0 \), and 2) \( □(\alpha \sim \beta) \geq \Delta 0 \).

In the case 1) by (7), (1) and (2) we get \( \neg(\alpha \sim \beta) \geq \Delta 0, \nabla \neg(\alpha \sim \beta) = 1 \), and \( \nabla(\alpha \sim \beta) = 0 \), which implies

\[
f_-(\alpha, \beta) = (\nabla \neg(\alpha \sim \beta) \& ((\neg \alpha \sim \beta) \sim \xi)) \lor (\nabla(\alpha \sim \beta) \& \alpha_i)
= (1 \& ((\neg \alpha \sim \beta) \sim \xi)) \lor (0 \& \alpha_i) = (\neg \alpha \sim \beta) \sim \xi \neq \xi,
\]
Thus the first case has already been examined.

Now consider the second case, when \( □x \geq \Delta 0 \). Again, since \( \neg \alpha \neq \beta \) by (1), (2) and (6) we obtain \( \neg(\alpha \sim \beta) = 0, \nabla \neg(\alpha \sim \beta) = 0, \nabla(\alpha \sim \beta) = 1 \). Then,

\[
f_-(\alpha, \beta) = (\nabla \neg(\alpha \sim \beta) \& ((\neg \alpha \sim \beta) \& \xi)) \lor (\nabla(\alpha \sim \beta) \& \alpha_i)
= (0 \& ((\neg \alpha \sim \beta) \& \xi)) \lor (1 \& \alpha_i) = \alpha_i \neq \xi.
\]
□
Proposition 4. Let arbitrary $\alpha, \beta \in M^*$ be such that $\Delta \alpha = \beta$. Then
\[ f_{\Delta}(\alpha, \beta) = \eta. \]

Proof. Since $\Delta \alpha \geq 0$ and $\Delta \alpha = \beta$ we have $\square \beta \geq \Delta 0$, $\Delta \alpha \sim \beta = 1$ and by (1) we get $\nabla \beta = 1$, $\neg \nabla \beta = 0$. These ones imply the following relations:
\[
 f_{\Delta}(\alpha, \beta) = (\nabla \beta \& ((\Delta \alpha \sim \beta) \sim \eta)) \lor (\neg \nabla \beta \& \alpha_i) = \nabla \beta = 1 \lor \alpha_i = 1 \sim \eta = \eta.
\]
\[ \square \]

Proposition 5. Let arbitrary $\alpha, \beta \in M^*$ be such that $\Delta \alpha \neq \beta$. Then
\[ f_{\Delta}(\alpha, \beta) \neq \eta. \]

Proof. We consider 2 cases: 1) $\square \beta = 0$, and 2) $\square \beta \geq \Delta 0$.

Suppose $\square \beta = 0$. In view of (2) we have $\nabla \beta = 0$ and $\neg \nabla \beta = 1$. Subsequently,
\[
 f_{\Delta}(\alpha, \beta) = (\nabla \beta \& ((\Delta \alpha \sim \beta) \sim \eta)) \lor (\neg \nabla \beta \& \alpha_i) = \nabla \beta = 0 \lor \alpha_i = \alpha_i \neq \eta.
\]

Suppose now $\square \beta \geq \Delta 0$. Let us note $\Delta \alpha \sim \beta \neq 1$. Then considering (1) we get
\[
 f_{\Delta}(\alpha, \beta) = (\nabla \beta \& ((\Delta \alpha \sim \beta) \sim \eta)) \lor (\neg \nabla \beta \& \alpha_i) = (\nabla \beta \& (\Delta \alpha \sim \beta \sim \eta)) \lor (\neg \nabla \beta \& \alpha_i) = (\Delta \alpha \sim \beta) \sim \eta \neq \eta.
\]
\[ \square \]

Proposition 6. Let arbitrary $\alpha \in M^*$. Then
\[ f_{\neg}(\alpha, \alpha) = \alpha_i. \]

Proof. Let us calculate $f_{\neg}(\alpha, \alpha)$. By (5) we obtain immediately:
\[
 f_{\neg}(\alpha, \alpha) = (\nabla \neg(\alpha \sim \alpha) \& ((\neg \alpha \sim \alpha) \& \xi)) \lor (\nabla(\alpha \sim \alpha) \& \alpha_i) = (\nabla(\alpha \sim \alpha) \& \alpha_i) = \alpha_i.
\]
\[ \square \]

Proposition 7. Let arbitrary $\alpha \in M^*$ and $\square \alpha = 0$. Then
\[ f_{\Delta}(\alpha, \alpha) = \alpha_i. \]

Proof. Taking into account (2) we have
\[
 f_{\Delta}(\alpha, \alpha) = (\nabla \alpha \& ((\Delta \alpha \sim \alpha) \sim \eta)) \lor (\neg \nabla \alpha \& \alpha_i) = (\nabla \alpha \& (\Delta \alpha \sim \alpha \sim \eta)) \lor (\neg \nabla \alpha \& \alpha_i) = 0 \lor \alpha_i = \alpha_i.
\]
\[ \square \]
4 Important properties of some primitive positive clones

Consider an arbitrary value \( i, i = 1, 2, \ldots \). Let \( K_i \) be the primitive positive clone of \( \mathcal{M}^* \) consisting of all functions of \( \mathcal{M}^* \) which preserve the relation \( x = -\Delta^i 0 \) on \( \mathcal{M}^* \). For example, \( K_1 \) is defined by the relation \( x = (0, 1, 1, \ldots) \).

Remark 2. The functions \( \square x, x \& y, x \lor y, -\Delta^i 0 \in K_i \), and \( -x, \Delta x \notin K_i \).

Remark 3. Since \( K_i \) is a primitive positive clone it follows from the above statement the functions \( -x \) and \( \Delta x \) are not primitive positive definable in \( \mathcal{M}^* \), so \( T(\mathcal{M}^*) \not\subseteq K_i \) and thus the clone \( K_i \) is not complete in \( \mathcal{M}^* \).

Remark 4. By Propositions 6 and 7 we have the earlier defined functions \( f_{-}(x, y) \) and \( f_{\Delta}(x, y) \) are in \( K_i \).

Lemma 1. Suppose an arbitrary \( f(x_1, \ldots, x_k) \in T(\mathcal{M}^*) \) and \( f \notin K_i \). Then the functions \( \Delta x \) and \( -x \) are primitive positive definable via functions of \( K_i \cup \{ f(x_1, \ldots, x_k) \} \).

Proof. Let us note since \( f \notin K_i \) we have \( f(-\Delta^i 0, \ldots, -\Delta^i 0) \neq -\Delta^i 0 \). Now consider the next term operations \( f'_{\Delta} \) and \( f'_{\Delta} \) defined by terms (10) and (11):

\[
\begin{align*}
(\nabla - (x \sim y) & \lor ((-x \sim y) \sim f(-\Delta^i 0, \ldots, -\Delta^i 0))) \lor (\nabla (x \sim y) \& -\Delta^i 0) \\
(\nabla y & \lor ((\Delta x \sim y) \sim f(-\Delta^i 0, \ldots, -\Delta^i 0))) \lor (-\nabla y \& -\Delta^i 0)
\end{align*}
\]

(10) and (11)

and examine the primitive positive formulas containing only functions from \( K_i \cup \{ f \} \):

\[
(f'_{\Delta}(x, y) = f(-\Delta^i 0, \ldots, -\Delta^i 0)) \text{ and } (f'_{\Delta}(x, y) = f(-\Delta^i 0, \ldots, -\Delta^i 0)).
\]

Let us note by Propositions 2 and 3 we have \( (-x = y) \) if and only if \( (f'_{\Delta}(x, y) = f(-\Delta^i 0, \ldots, -\Delta^i 0)) \) and according to Propositions 4 and 5 we get \( (\Delta x = y) \) if and only if \( (f'_{\Delta}(x, y) = f(-\Delta^i 0, \ldots, -\Delta^i 0)).

Lemma is proved. \( \square \)

5 Main result

Theorem 1. There are infinitely many maximal primitive positive clones in the diagonalizable algebra \( \mathcal{M}^* \).

Proof. The theorem is based on the example of an infinite family of maximal primitive positive clones presented below.

Example 1. The classes \( K_1, K_2, \ldots \) of term operations of \( T(\mathcal{M}^*) \), which preserve on algebra \( \mathcal{M}^* \) the corresponding relations \( x = -\Delta 0, x = -\Delta^2 0, \ldots \), constitute a numerable collection of maximal primitive positive clones in \( \mathcal{M}^* \).

Really, it is known [9] that these classes of functions represent primitive positive clones. According to Remark 3 each clone \( K_i \) is not complete in \( \mathcal{M}^* \). In virtue of Lemma 1 these primitive positive clones are maximal. It remains to show these clones are different. The last thing is obvious since

\( -\Delta^i 0 \in K_j \) and \( -\Delta^i 0 \notin K_i \), when \( i \neq j \).

The theorem is proved. \( \square \)
6 Conclusions

We can consider the logic $L_{M^*}$ of $M^*$, which happens to be an extension of the propositional provability logic $GL$, and consider primitive positive classes of formulas $M_1, M_2, \ldots$ of the propositional provability calculus of $GL$ preserving on $M^*$ the corresponding relations $x = -\Delta^0, x = -\Delta^20, \ldots$.

**Theorem 2.** The classes of formulas $M_1, M_2, \ldots$ constitute an infinite collection of primitive positive classes of formulas in the extension $L_{M^*}$ of the propositional provability logic $GL$.

**Proof.** The statement of the theorem is just another formulation of the Theorem 1 above in terms of formulas of the calculus of $GL$, which follows the usual terminology of [3].

References


